Exercise 4

Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

(a) Evaluate $g(0)$ and $g(6)$.

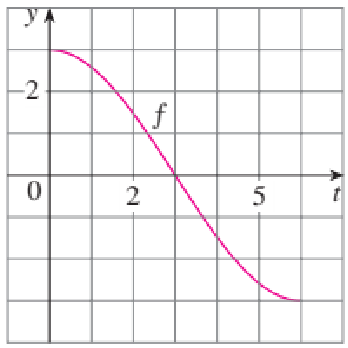
(b) Estimate $g(x)$ for $x = 1$, 2, 3, 4, and 5.

(c) On what interval is g increasing?

(d) Where does g have a maximum value?

(e) Sketch a rough graph of g.

(f) Use the graph in part (e) to sketch the graph of $g'(x)$. Compare with the graph of f.



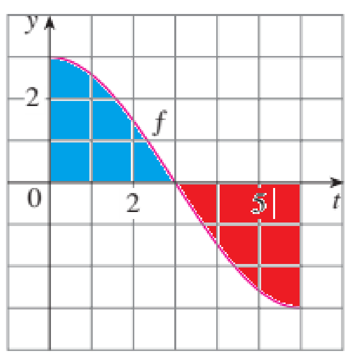
Solution

Part (a)

For $g(0)$, the limits of integration are the same, which makes the integral zero.

$$g(0) = \int_0^0 f(t)\,dt = 0$$

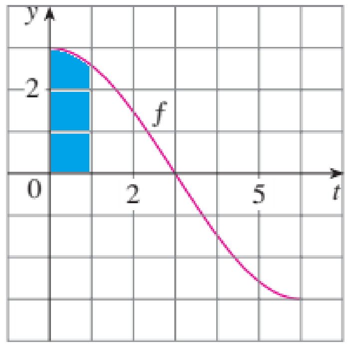
$g(6)$ is the area under the curve from $x = 0$ to $x = 6$.



Since there's as much area above the x-axis as there is below the x-axis, $g(6) = 0$.

Part (b)

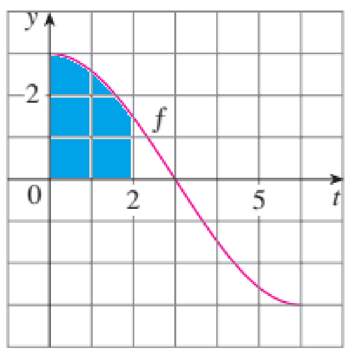
$g(1)$ is the area under the curve from $x = 0$ to $x = 1$.



Each square has an area of 1. The top-right quarter of the third square is missing, so

$$g(1) \approx 1 + 1 + 0.75 = 2.75.$$

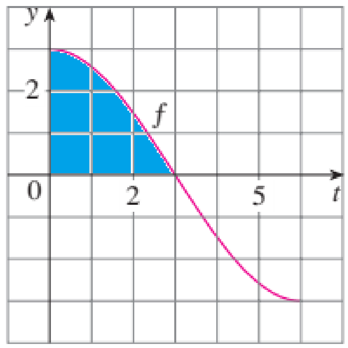
$g(2)$ is the area under the curve from $x = 0$ to $x = 2$.



The area from $x = 1$ to $x = 2$ consists of two squares roughly.

$$\begin{aligned} g(2) &= \int_0^2 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \\ &\approx 2.75 + (1 + 0.75 + 0.25) \\ &\approx 4.75 \end{aligned}$$

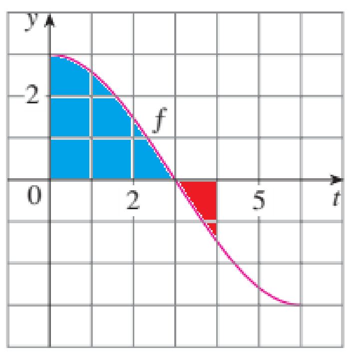
$g(3)$ is the area under the curve from $x = 0$ to $x = 3$.



The area from $x = 2$ to $x = 3$ consists of three quarters of a square roughly.

$$\begin{aligned}
g(3) &= \int_0^3 f(t)\,dt \\
&= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt \\
&\approx 2.75 + (1 + 0.75 + 0.25) + 0.75 \\
&\approx 5.5
\end{aligned}$$

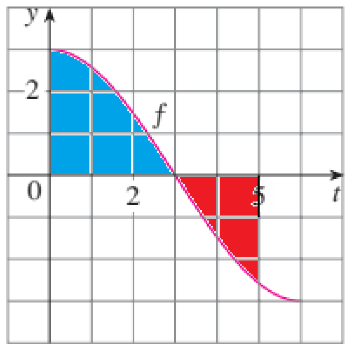
$g(4)$ is the area under the curve from $x = 0$ to $x = 4$.



The area from $x = 3$ to $x = 4$ is the same area as that from $x = 2$ to $x = 3$ but under the x-axis.

$$\begin{aligned}
g(4) &= \int_0^4 f(t)\,dt \\
&= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt \\
&\approx 2.75 + (1 + 0.75 + 0.25) + 0.75 + (-0.75) \\
&\approx 4.75
\end{aligned}$$

$g(5)$ is the area under the curve from $x = 0$ to $x = 5$.



The area from $x = 4$ to $x = 5$ is the same area as that from $x = 1$ to $x = 2$ but under the x-axis.

$$\begin{aligned} g(5) &= \int_0^5 f(t)\,dt \\ &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt + \int_4^5 f(t)\,dt \\ &\approx 2.75 + (1 + 0.75 + 0.25) + 0.75 + (-0.75) + (-1 - 0.75 - 0.25) \\ &\approx 2.75 \end{aligned}$$

Part (c)

g is increasing on the interval $0 < x < 3$ because the curve is above the x-axis here.

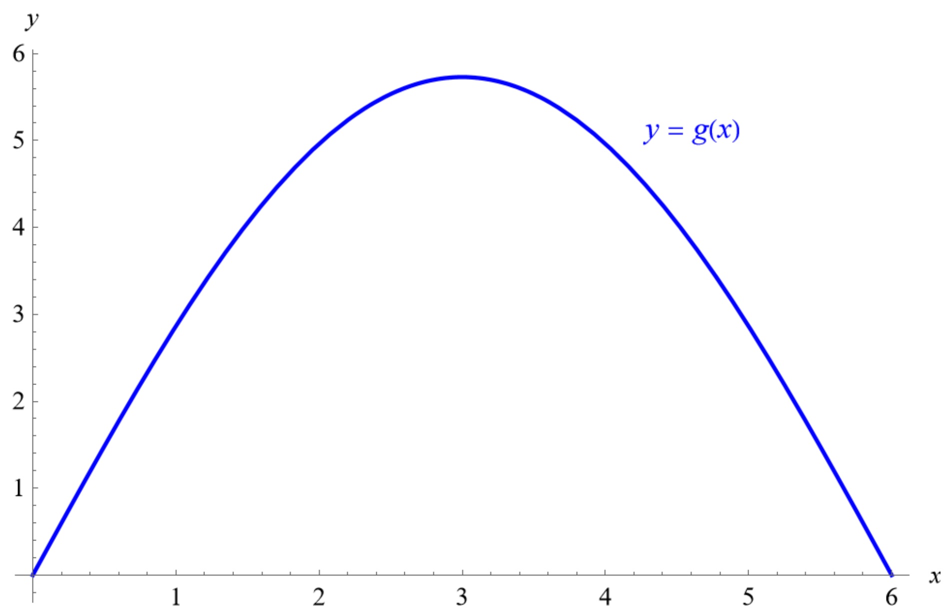
Part (d)

g has a maximum at $x = 3$ because that's where the curve goes below the x-axis.

$$g(3) \approx 5.5$$

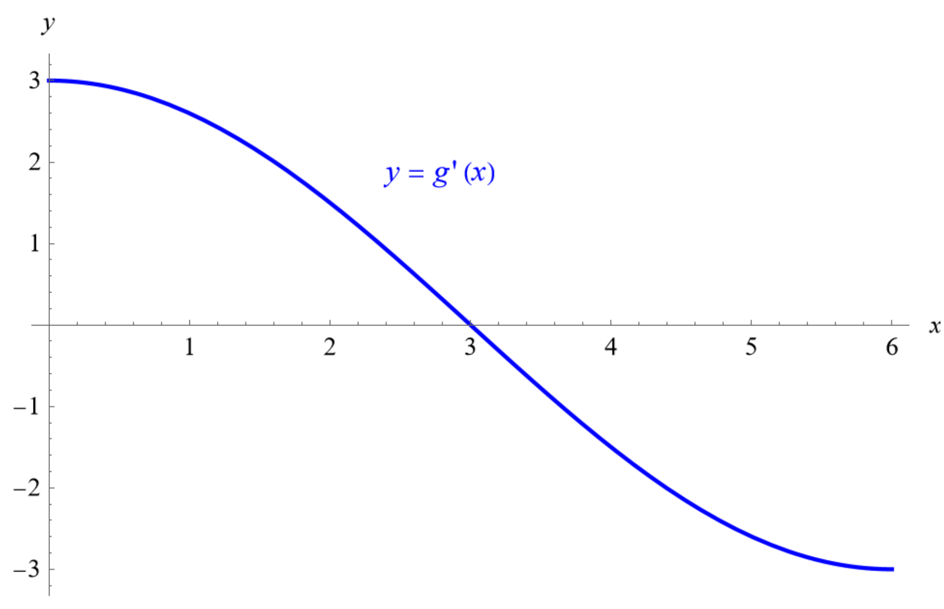
Part (e)

Below is a graph of $g(x)$ versus x for $0 \leq x \leq 6$.



Part (f)

Below is a graph of $g'(x)$ versus x for $0 \leq x \leq 6$.



$g'(x)$ and $f(x)$ have the same graph.

$$g'(x) = f(x)$$